5.31. (a)

$$H(z) = \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}, \quad \text{stable, so the ROC is } \frac{1}{2} < |z| < 3$$

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$Y(z) = X(z)H(z) = \frac{\frac{4}{5}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{5}}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \quad 1 < |z| < 3$$

$$y[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{5}(3)^n u[-n-1] - u[n]$$

(b) ROC includes $z = \infty$ so $h[n]$ is causal. Since both $h[n]$ and $x[n]$ are 0 for $n < 0$, we know that $y[n]$ is also 0 for $n < 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

$$Y(z) - \frac{7}{2}z^{-1}Y(z) + \frac{3}{2}z^{-2}Y(z) = z^{-2}X(z)$$

$$y[n] = x[n-2] + \frac{7}{2}y[n-1] - \frac{3}{2}y[n-2]$$

Since $y[n] = 0$ for $n < 0$, recursion can be done:

$$y[0] = 0, \quad y[1] = 0, \quad y[2] = 1$$

(c)

$$H_i(z) = \frac{1}{H(z)} = z^2 - \frac{7}{2}z + \frac{3}{2}, \quad \text{ROC: entire z-plane}$$

$$h_i[n] = \delta[n+2] - \frac{7}{2}\delta[n+1] + \frac{3}{2}\delta[n]$$

5.37.

$$X(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{5}z)}{(1-\frac{1}{6}z)} = \frac{6}{5}\frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-5z^{-1})}{(1-6z^{-1})}$$

$$\alpha^n x[n] \Leftrightarrow X(\alpha^{-1}z) = \frac{6}{5}\frac{(1-\frac{1}{2}\alpha z^{-1})(1-\frac{1}{4}\alpha z^{-1})(1-5\alpha z^{-1})}{(1-6\alpha z^{-1})}$$

A minimum phase sequence has all poles and zeros inside the unit circle.

$$|\alpha/2| < 1 \;\Rightarrow\; |\alpha| < 2$$
$$|\alpha/4| < 1 \;\Rightarrow\; |\alpha| < 4$$
$$|5\alpha| < 1 \;\Rightarrow\; |\alpha| < \frac{1}{5}$$
$$|6\alpha| < 1 \;\Rightarrow\; |\alpha| < \frac{1}{6}$$

Therefore, $\alpha^n x[n]$ is real and minimum phase iff α is real and $|\alpha| < \frac{1}{6}$.

5.39. All zeros inside the unit circle means the sequence is minimum phase. Since

$$\sum_{n=0}^{M} |h_{min}[n]|^2 \geq \sum_{n=0}^{M} |h[n]|^2$$

is true for all M, we can use $M = 0$ and just compute $h^2[0]$. The largest result will be the minimum phase sequence.

A	B	C	D	E	F	G	H
44.5	28.4	1.8	2.8	1.8	177.7	113.8	7.1

The answer is F.

5.53. For all of the following we know that the poles and zeros are real or occur in complex conjugate pairs since each impulse response is real. Since they are causal we also know that none have poles at infinity.

(a)
- Since $h_1[n]$ is real there are complex conjugate poles at $z = 0.9e^{\pm j\pi/3}$.
- If $x[n] = u[n]$

$$Y(z) = H_1(z)X(z) = \frac{H_1(z)}{1 - z^{-1}}$$

We can perform a partial fraction expansion on $Y(z)$ and find a term $(1)^n u[n]$ due to the pole at $z = 1$. Since $y[n]$ eventually decays to zero this term must be cancelled by a zero. Thus, the filter must have a zero at $z = 1$.

- The length of the impulse response is infinite.

(b)
- Linear phase and a real impulse response implies that zeros occur at conjugate reciprocal locations so there are zeros at $z = z_1, 1/z_1, z_1^*, 1/z_1^*$ where $z_1 = 0.8e^{j\pi/4}$.
- Since $h_2[n]$ is both causal and linear phase it must be a Type I, II, III, or IV FIR filter. Therefore the filter's poles only occur at $z = 0$.
- Since the $\arg\{H_2(e^{j\omega})\} = -2.5\omega$ we can narrow down the filter to a Type II or Type IV filter. This also tells us that the length of the impulse response is 6 and that there are 5 zeros. Since the number of poles always equal the number of zeros, we have 5 poles at $z = 0$.
- Since $20\log|H_2(e^{j0})| = -\infty$ we must have a zero at $z = 1$. This narrows down the filter type even more from a Type II or Type IV filter to just a Type IV filter.

With all the information above we can determine $H_2(z)$ completely (up to a scale factor)

$$H_2(z) = A(1 - z^{-1})(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 1.25e^{j\pi/4}z^{-1})(1 - 1.25e^{-j\pi/4}z^{-1})$$

(c) Since $H_3(z)$ is allpass we know the poles and zeros occur in conjugate reciprocal locations. The impulse response is infinite and in general looks like

$$H_3(z) = \frac{(z^{-1} - 0.8e^{j\pi/4})(z^{-1} - 0.8e^{-j\pi/4})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} H_{ap}(z)$$

6.23. Causal LTI system with system function:

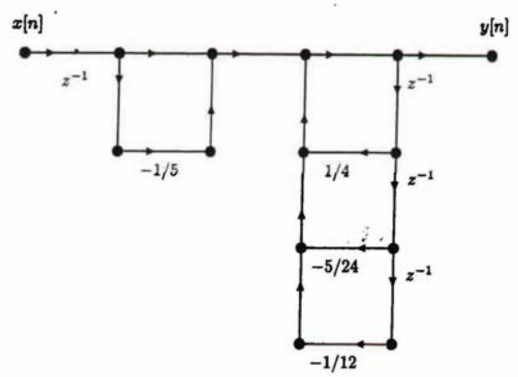
$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}.$$

(a) (i) Direct form I.

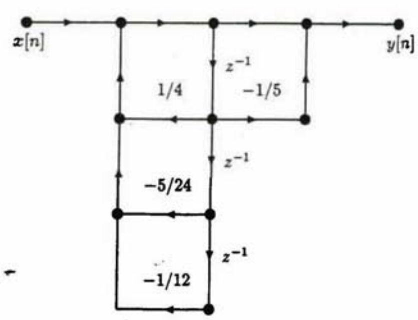
$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z(-3)}$$

so

$$b_0 = 1\ ,\ b_1 = -\frac{1}{5}\ \text{and}\ a_1 = \frac{1}{4}\ ,\ a_2 = -\frac{5}{24}\ ,\ a_3 = -\frac{1}{12}.$$



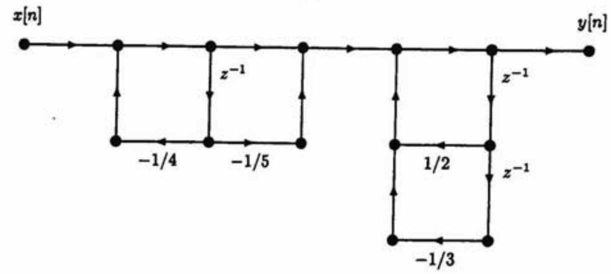
(ii) Direct form II.



$$H(z) = \left(\frac{1 - \frac{1}{5}z^{-1}}{1 + \frac{1}{4}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}}\right).$$

So

$$b_{01} = 1\ ,\ b_{11} = -\tfrac{1}{5}\ ,\ b_{21} = 0\ ,$$
$$b_{02} = 1\ ,\ b_{12} = 0\ ,\ b_{22} = 0 \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$

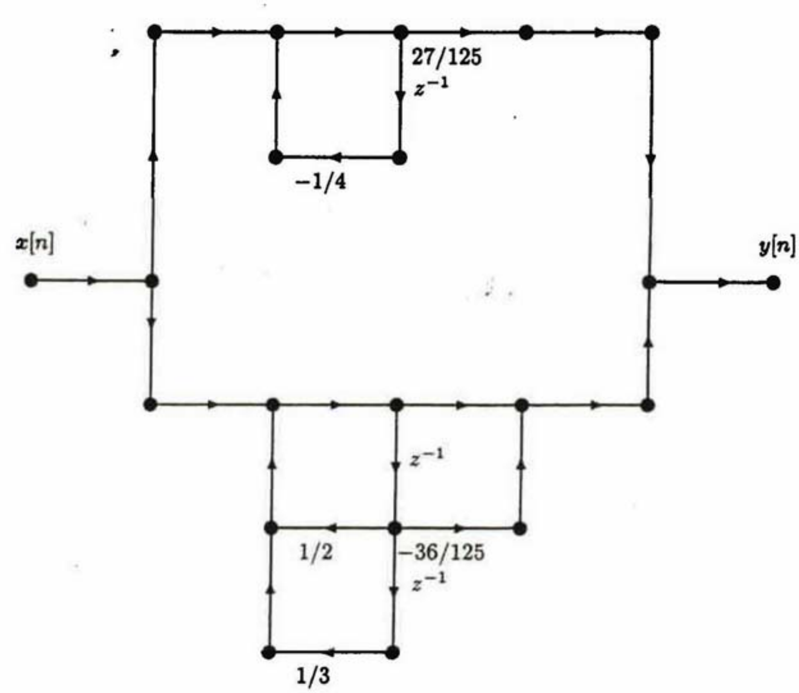


(iv) Parallel form using first and second order direct form II sections.
We can rewrite the transfer function as:

$$H(z) = \frac{\frac{27}{125}}{1 + \frac{1}{4}z^{-1}} + \frac{\frac{98}{125} - \frac{36}{125}z^{-1}}{1 - \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}}.$$

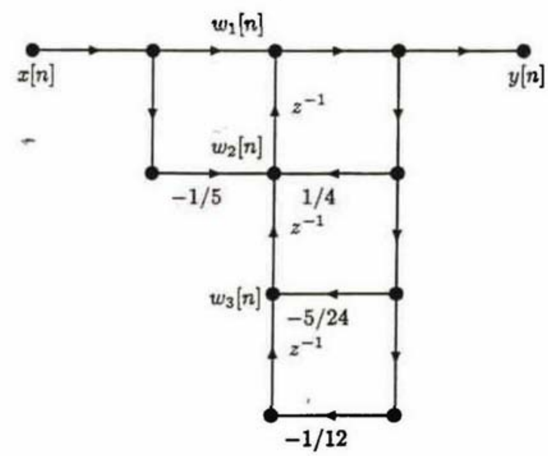
So

$$e_{01} = \tfrac{27}{125}\ ,\ e_{11} = 0\ ,$$
$$e_{02} = \tfrac{98}{125}\ ,\ e_{12} = -\tfrac{36}{125}\ ,\ \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$



(v) Transposed direct form II

We take the direct form II derived in part (ii) and reverse the arrows as well as exchange the input and output. Then redrawing the flow graph, we get:



(b) To get the difference equation for the flow graph of part (v) in (a), we first define the intermediate variables: $w_1[n]$, $w_2[n]$ and $w_3[n]$. We have:

$$
\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= \frac{1}{4}y[n] + w_3[n-1] - \frac{1}{5}x[n] \\
(3)\ w_3[n] &= -\frac{5}{24}y[n] - \frac{1}{12}y[n-1] \\
(4)\quad y[n] &= w_1[n].
\end{aligned}
$$

Combining the above equations, we get:

$$y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n-1].$$

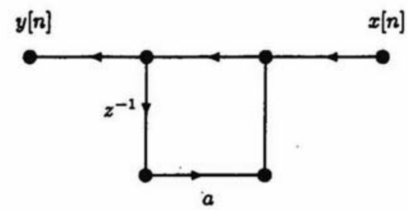
Taking the Z-transform of this equation and combining terms, we get the following transfer function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

which is equal to the initial transfer function.

6.24. (a)

$$H(z) = \frac{1}{1 - az^{-1}}$$

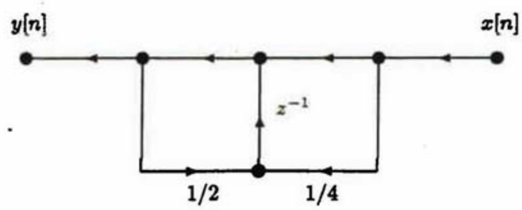


$$y[n] = x[n] + ay[n-1]$$

$$H_T(z) = \frac{1}{1 - az^{-1}} = H(z)$$

(b)

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

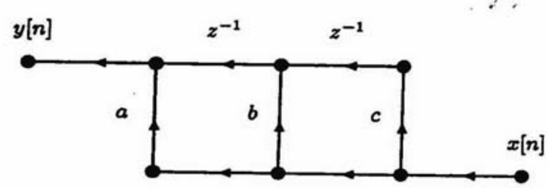


$$y[n] = x[n] + \frac{1}{4}x[n-1] + \frac{1}{2}y[n-1]$$

$$H_T(z) = \frac{1+\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} = H(z)$$

(c)

$$H(z) = a + bz^{-1} + cz^{-2}$$

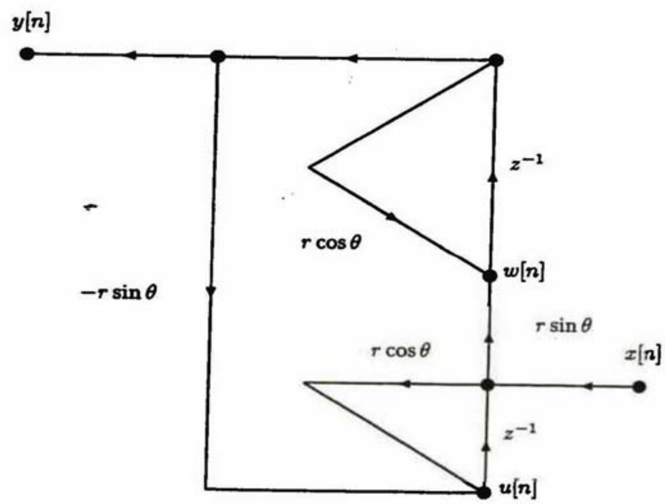


$$y[n] = ax[n] + bx[n-1] + cx[n-2]$$

$$H_T(z) = a + bz^{-1} + cz^{-2} = H(z)$$

(d)

$$H(z) = \frac{r\sin\theta z^{-1}}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$$



$$\begin{aligned}
V &= X + z^{-1}U \\
U &= r\cos\theta V - r\sin\theta Y \\
W &= r\sin\theta V + r\cos\theta z^{-1}W \\
Y &= z^{-1}W \\
\Longrightarrow \frac{Y}{X} &= H_T(z) \\
&= \frac{r\sin\theta z^{-1}}{1-2r\cos\theta z^{-1} + r^2z^{-2}} \\
&= H(z).
\end{aligned}$$