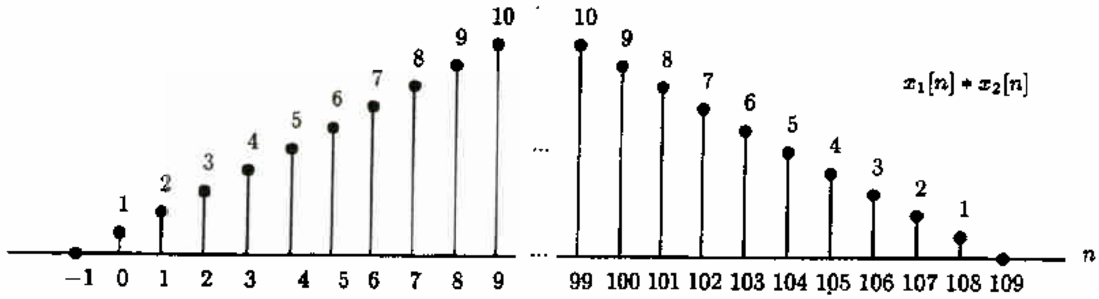
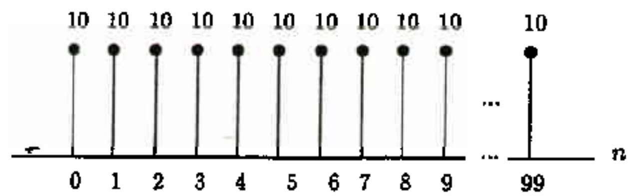
8.27. (a) The linear convolution, $x_1[n] * x_2[n]$ is a sequence of length $100 + 10 - 1 = 109$.

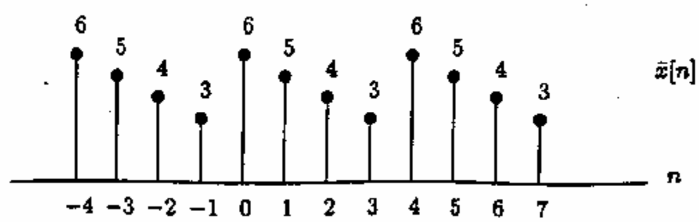


(b) The circular convolution, $x_1[n]\,\text{(100)}\,x_2[n]$, can be obtained by aliasing the first 9 points of the linear convolution above:

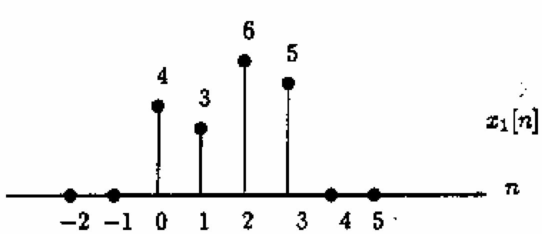


(c) Since $N \geq 109$, the circular convolution $x_1[n]\,\text{(110)}\,x_2[n]$ will be equivalent to the linear convolution of part (a).

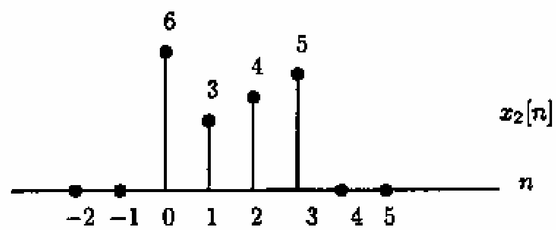
8.28. We may approach this problem in two ways. First, the notion of modulo arithmetic may be simplified if we utilize the implied periodic extension. That is, we redraw the original signal as if it were periodic with period $N = 4$. A few periods are sufficient:



To obtain $x_1[n] = x[((n-2))_4]$, we shift by two (to the right) and only keep those points which lie in the original domain of the signal (i.e. $0 \leq n \leq 3$):



To obtain $x_2[n] = x[((-n))_4]$, we fold the pseudo-periodic version of $x[n]$ over the origin (time-reversal), and again we set all points outside $0 \leq n \leq 3$ equal to zero. Hence,



Note that $x[((0))_4] = x[0]$, etc.

In the second approach, we work with the given signal. The signal is confined to $0 \le n \le 3$; therefore, the circular nature must be maintained by picturing the signal on the circumference of a cylinder.

8.32. We have a finite-length sequence, $x[n]$ with $N = 8$. Suppose we interpolate by a factor of two. That is, we wish to double the size of $x[n]$ by inserting zeros at all odd values of n for $0 \le n \le 15$.

Mathematically,

$$y[n] = \begin{cases} x[n/2], & n \text{ even}, \quad 0 \le n \le 15 \\ 0; & n \text{ odd}, \end{cases}$$

The 16-pt. DFT of $y[n]$:

$$\begin{aligned} Y[k] &= \sum_{n=0}^{15} y[n] W_{16}^{kn}, \quad 0 \le k \le 15 \\ &= \sum_{n=0}^{7} x[n] W_{16}^{2kn} \end{aligned}$$

Recall, $W_{16}^{2kn} = e^{j(2\pi/16)(2k)n} = e^{-j(2\pi/8)kn} = W_8^{kn}$,

$$Y[k] = \sum_{n=0}^{7} x[n] W_8^{kn}, \quad 0 \le k \le 15$$

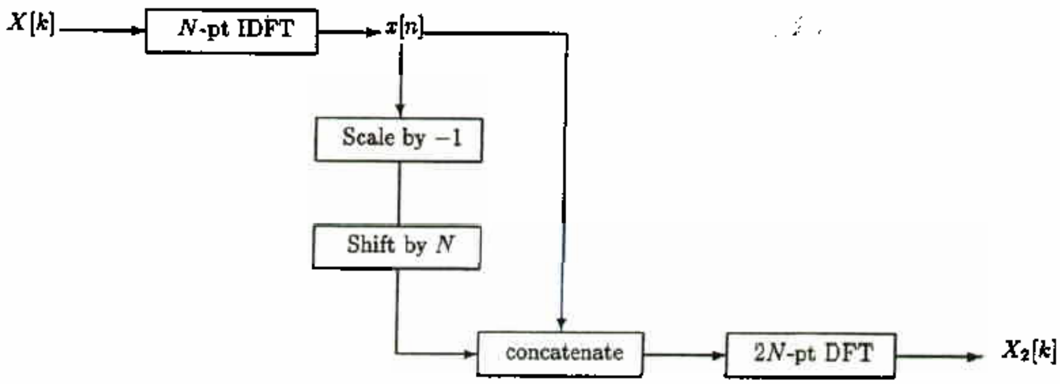
Therefore, the 16-pt. DFT of the interpolated signal contains two copies of the 8-pt. DFT of $x[n]$. This is expected since $Y[k]$ is now periodic with period 8 (see problem 8.1). Therefore, the correct choice is C.

As a quick check, $Y[0] = X[0]$.

8.33. (a) Since

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ -x[n-N], & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$$

If $X[k]$ is known, $x_2[n]$ can be constructed by :

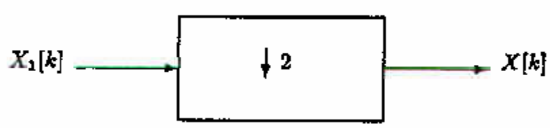


(b) To obtain $X[k]$ from $X_1[k]$, we might try to take the inverse DFT (2N-pt) of $X_1[k]$, then take the N-pt DFT of $x_1[n]$ to get $X[k]$.

However, the above approach is highly inefficient. A more reasonable approach may be achieved if we examine the DFT analysis equations involved. First,

$$\begin{aligned} X_1[k] &= \sum_{n=0}^{2N-1} x_1[n] W_{2N}^{kn}, \qquad 0 \le k \le (2N-1) \\ &= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{(k/2)n}, \quad 0 \le k \le (N-1) \\ X_1[k] &= X[k/2], \quad 0 \le k \le (N-1) \end{aligned}$$

Thus, an easier way to obtain $X[k]$ from $X_1[k]$ is simply to decimate $X_1[k]$ by two.



8.36. We have the finite-length sequence:

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(i) Suppose we perform the 5-pt DFT:

$$X[k] = 2 + W_5^k + W_5^{3k}, \quad 0 \le k \le 5$$

where $W_5^k = e^{-j(\frac{2\pi}{5})k}$.

(ii) Now, we square the DFT of $x[n]$:

$$\begin{aligned} Y[k] &= X^2[k] \\ &= 2 + 2W_5^k + 2W_5^{3k} \\ &\quad + 2W_5^k + W_5^{2k} + W_5^{5k} \\ &\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \quad 0 \le k \le 5 \end{aligned}$$

Using the fact $W_5^{5k} = W_5^0 = 1$ and $W_5^{6k} = W_5^k$

$$Y[k] = 3 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + W_5^{4k}, \quad 0 \le k \le 5$$

(a) By inspection,

$$y[n] = 3\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-3] + \delta[n-4], \quad 0 \le n \le 5$$

(b) This procedure performs the autocorrelation of a real sequence. Using the properties of the DFT, an alternative method may be achieved with convolution:

$$y[n] = \text{IDFT}\{X^2[k]\} = x[n] * x[n]$$

The IDFT and DFT suggest that the convolution is circular. Hence, to ensure there is no aliasing, the size of the DFT must be $N \ge 2M - 1$ where M is the length of $x[n]$. Since $M = 3, N \ge 5$.

9.21. (a) Assume $x[n] = 0$, for $n < 0$ and $n > N - 1$. From the figure, we see that

$$y_k[n] = x[n] + W_N^k y_k[n-1]$$

Starting with $n = 0$, and iterating this recursive equation, we find

$$\begin{aligned} y_k[0] &= x[0] \\ y_k[1] &= x[1] + W_N^k x[0] \\ y_k[2] &= x[2] + W_N^k x[1] + W_N^{2k} x[0] \\ &\vdots \\ y_k[N] &= x[N] + W_N^k x[N-1] + \cdots + W_N^{k(N-1)} x[1] + W_N^{kN} x[0] \\ &= 0 + \sum_{\ell=0}^{N-1} W_N^{k(N-\ell)} x[\ell] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{N-1} W_N^{-k\ell} x[\ell] \\
&= \sum_{\ell=0}^{N-1} x[\ell] W_N^{(N-k)\ell} \\
&= X[N-k]
\end{aligned}
$$

(b) Using the figure, we find the system function $Y_k(z)$.

$$
\begin{aligned}
Y_k(z) &= X(z)\frac{1 - W_N^{-k} z^{-1}}{1 - 2z^{-1}\cos(\frac{2\pi k}{N}) + z^{-2}} \\
&= X(z)\frac{1 - W_N^{-k} z^{-1}}{(1 - W_N^{-k} z^{-1})(1 - W_N^{k} z^{-1})} \\
&= \frac{X(z)}{1 - W_N^{k} z^{-1}}
\end{aligned}
$$

Therefore, $y_k[n] = x[n] + W_N^k y_k[n-1]$. This is the same difference equation as in part (a).

9.30. (a) Note that we can write the even-indexed values of $X[k]$ as $X[2k]$ for $k = 0, \ldots, (N/2) - 1$. From the definition of the DFT, we find

$$
\begin{aligned}
X[2k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi(2k)n/N} \\
&= \sum_{n=0}^{N/2-1} x[n] e^{-j\frac{2\pi}{(N/2)}kn} \\
&\quad + \sum_{n=0}^{N/2-1} x[n + (N/2)] e^{-j\frac{2\pi}{(N/2)}kn} e^{-j\frac{2\pi}{(N/2)}(N/2)k} \\
&= \sum_{n=0}^{N/2-1} (x[n] + x[n + (N/2)]) e^{-j\frac{2\pi}{(N/2)}kn} \\
&= Y[k]
\end{aligned}
$$

Thus, the algorithm produces the desired results.

(b) Taking the M-point DFT $Y[k]$, we find

$$
\begin{aligned}
Y[k] &= \sum_{n=0}^{M-1} \sum_{r=-\infty}^{\infty} x[n + rM] e^{-j2\pi kn/M} \\
&= \sum_{r=-\infty}^{\infty} \sum_{n=0}^{M-1} x[n + rM] e^{-j2\pi k(n+rM)/M} e^{j2\pi(rM)k/M}
\end{aligned}
$$

Let $l = n + rM$. This gives

$$
\begin{aligned}
Y[k] &= \sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi kl/M} \\
&= X(e^{j2\pi k/M})
\end{aligned}
$$

Thus, the result from Part (a) is a special case of this result if we let $M = N/2$. In Part (a), there are only two r terms for which $y[n]$ is nonzero in the range $n = 0, \ldots, (N/2) - 1$.

(c) We can write the odd-indexed values of $X[k]$ as $X[2k+1]$ for $k = 0, \ldots, (N/2)-1$. From the definition of the DFT, we find

$$\begin{aligned}
X[2k+1] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi(2k+1)n/N} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j2\pi n/N} e^{-j2\pi(2k)n/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n] e^{-j2\pi n/N} e^{-j\frac{2\pi}{(N/2)}kn} + \sum_{n=0}^{(N/2)-1} x[n+(N/2)] e^{-j2\pi[n+(N/2)]/N} e^{-j\frac{2\pi}{N/2}k(n+(N/2))} \\
&= \sum_{n=0}^{(N/2)-1} \left[(x[n] - x[n+(N/2)]) e^{-j\frac{2\pi}{N}n} \right] e^{-j\frac{2\pi}{(N/2)}kn}
\end{aligned}$$

Let

$$y[n] = \begin{cases} (x[n] - x[n+(N/2)]) e^{-j(2\pi/N)n}, & 0 \le n \le (N/2)-1 \\ 0, & \text{otherwise} \end{cases}$$

Then $Y[k] = X[2k+1]$. Thus, The algorithm for computing the odd-indexed DFT values is as follows.

step 1: Form the sequence

$$y[n] = \begin{cases} (x[n] - x[n+(N/2)]) e^{-j(2\pi/N)n}, & 0 \le n \le (N/2)-1 \\ 0, & \text{otherwise} \end{cases}$$

step 2: Compute the $N/2$ point DFT of $y[n]$, yielding the sequence $Y[k]$.

step 3: The odd-indexed values of $X[k]$ are then $X[k] = Y[(k-1)/2]$, $k = 1, 3, \ldots, N-1$.

9.34. This problem asks that we find eight equally spaced *inverse* DFT coefficients using the chirp transform algorithm. The book derives the algorithm for the forward DFT. However, with some minor tweaking, it is easy to formulate an inverse DFT. First, we start with the inverse DFT relation

$$\begin{aligned}
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \\
x[n_\ell] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi n_\ell k/N}
\end{aligned}$$

Next, we define

$$\begin{aligned} \Delta n &= 1 \\ n_\ell &= n_0 + \ell\Delta n \end{aligned}$$

where $\ell = 0, \ldots, 7$. Substituting this into the equation above gives

$$x[n_\ell] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_0 k/N}e^{j2\pi\ell\Delta n k/N}$$

Defining

$$W = e^{-j2\pi\Delta n/N}$$

we find

$$x[n_\ell] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_0 k/N}W^{-\ell k}$$

Using the relation

$$\ell k = \frac{1}{2}[\ell^2 + k^2 - (k-\ell)^2]$$

we get

$$x[n_\ell] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_0 k/N}W^{-\ell^2/2}W^{-k^2/2}W^{(k-\ell)^2/2}$$

Let

$$G[k] = X[k]e^{j2\pi n_0 k/N}W^{-k^2/2}$$

Then,

$$x[n_\ell] = \frac{1}{N}W^{-\ell^2/2}\left(\sum_{k=0}^{N-1} G[k]W^{(k-\ell)^2/2}\right)$$

From this equation, it is clear that the inverse DFT can be computed using the chirp transform algorithm. All we need to do is replace n by k, change the sign of each of the exponential terms, and divide by a factor of N. Therefore,

$$\begin{aligned} m_1[k] &= e^{j2\pi k n_0/N}W^{-k^2/2} \\ m_2[k] &= W^{-k^2/2} \\ h[k] &= \frac{1}{N}W^{k^2/2} \end{aligned}$$

Using this system with $n_0 = 1020$, and $\ell = 0, \ldots, 7$ will result in a sequence $y[n]$ which will contain the desired samples, where

$$\begin{aligned} y[0] &= x[1020] \\ y[1] &= x[1021] \\ y[2] &= x[1022] \\ y[3] &= x[1023] \\ y[4] &= x[0] \\ y[5] &= x[1] \\ y[6] &= x[2] \\ y[7] &= x[3] \end{aligned}$$